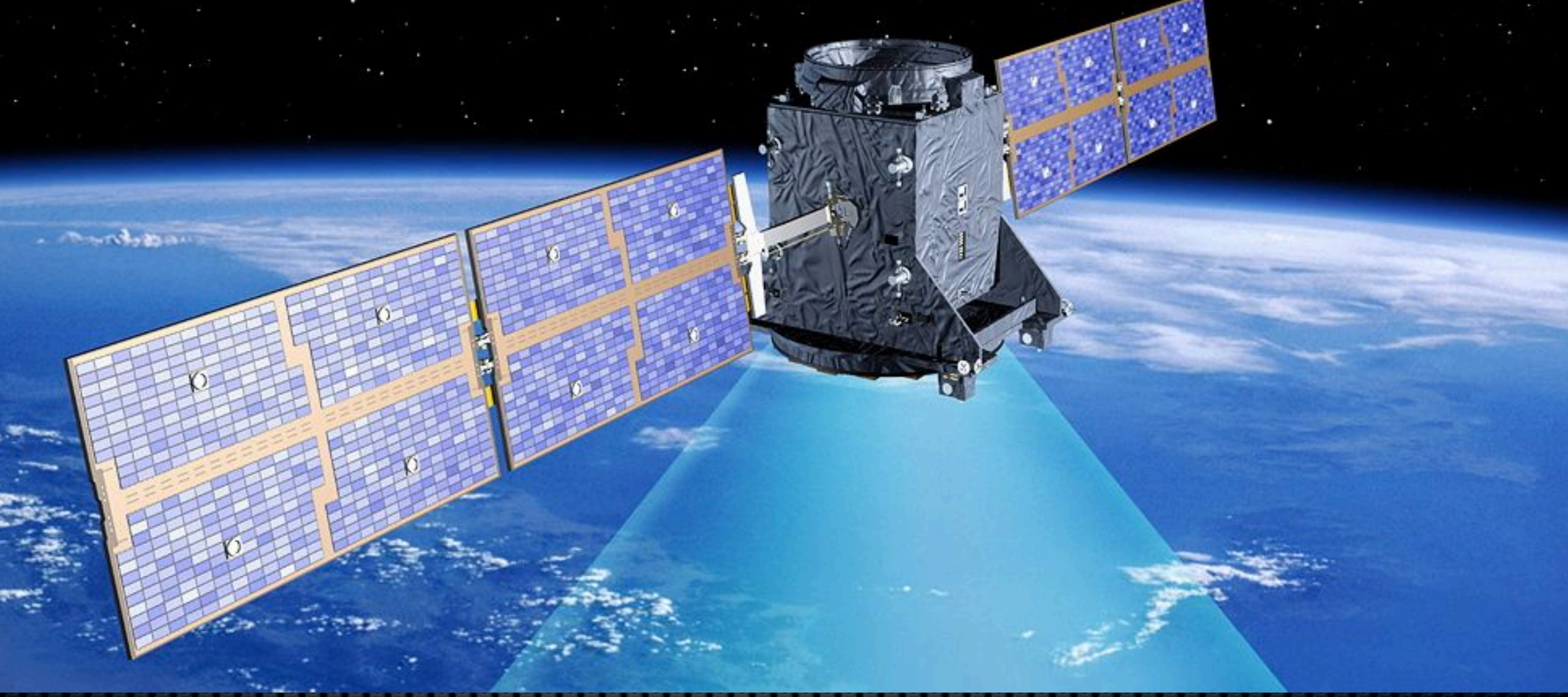


Large-Angle Optimal Low-Thrust Transfer of GEO Satellites Applying a Quaternion-Based Formulation

Paola Libraro, N. Jeremy Kasdin
Princeton University

All-Electric GEO Satellites

WHY?

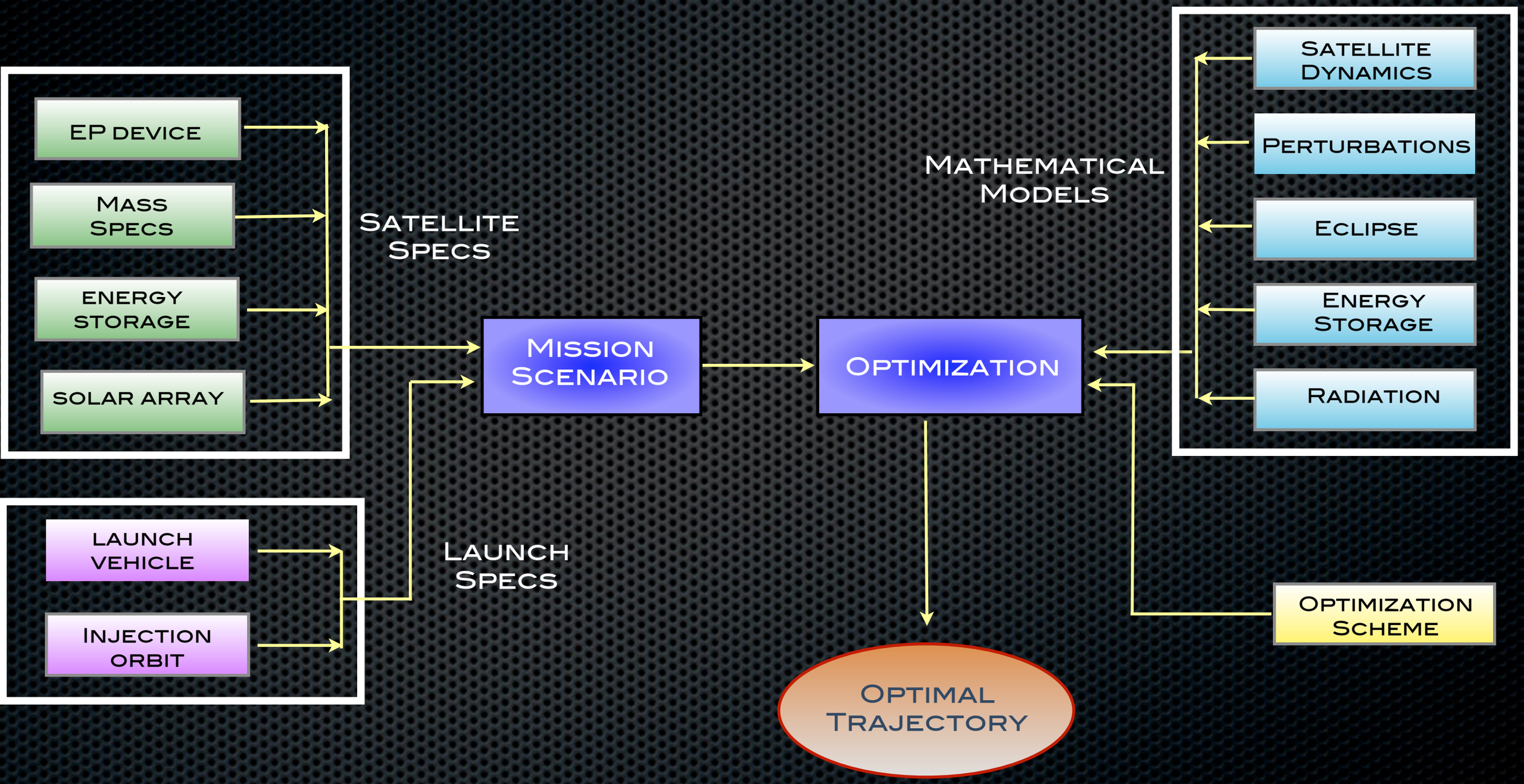
Lower launch costs ← Reduce weight and size of current satellites → More room for payload

Overview

1. Problem statement
2. Quaternion-based formulation
3. Integration & Testing
4. Optimization
5. Optimization results
   - Planar Maneuvers
   - Plane-Change Transfers
6. Concluding remarks

1. Electric Orbit-Raising Problem

Research Approach

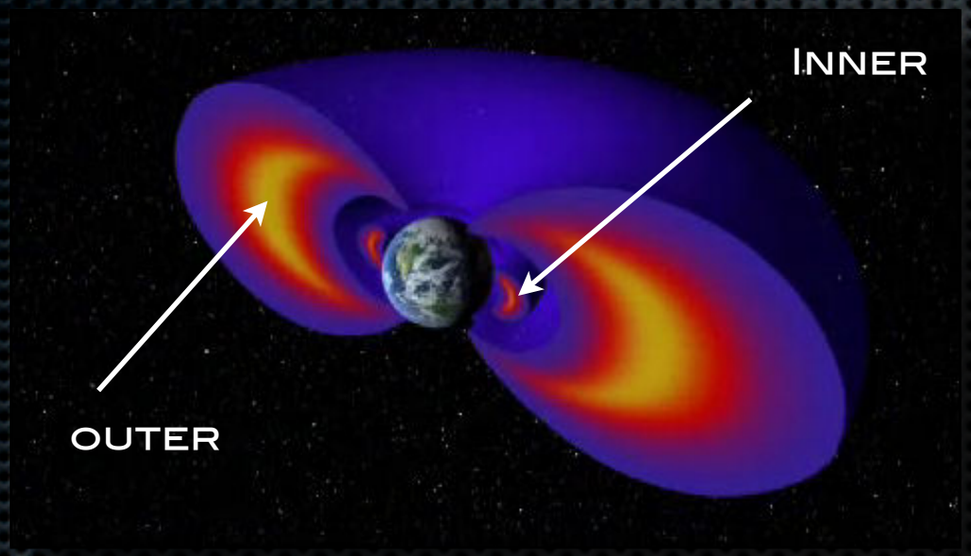


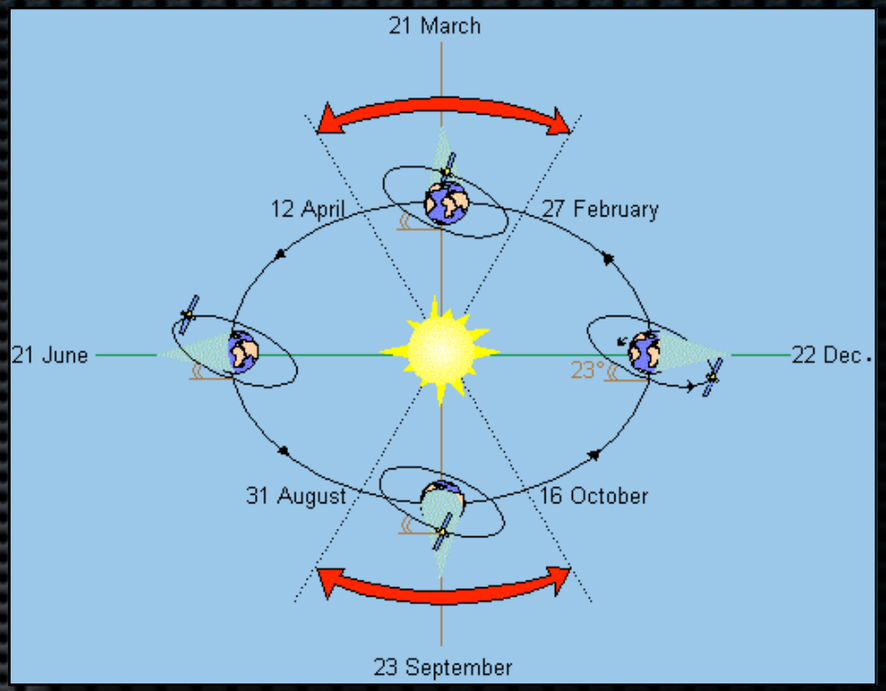
1. Problem Statement

Challenges

Van Allen radiation belt

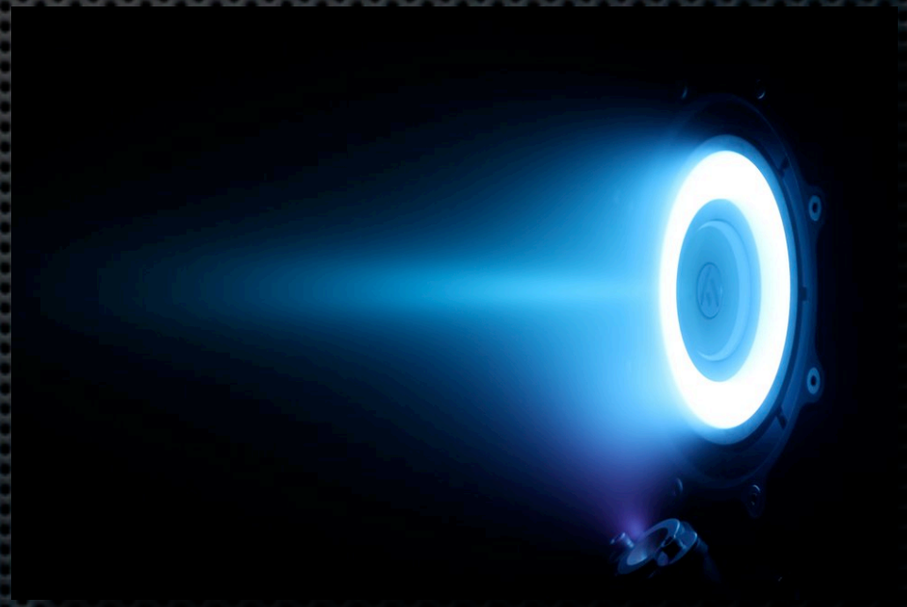
High-energy particles cause radiation damage to the solar cells





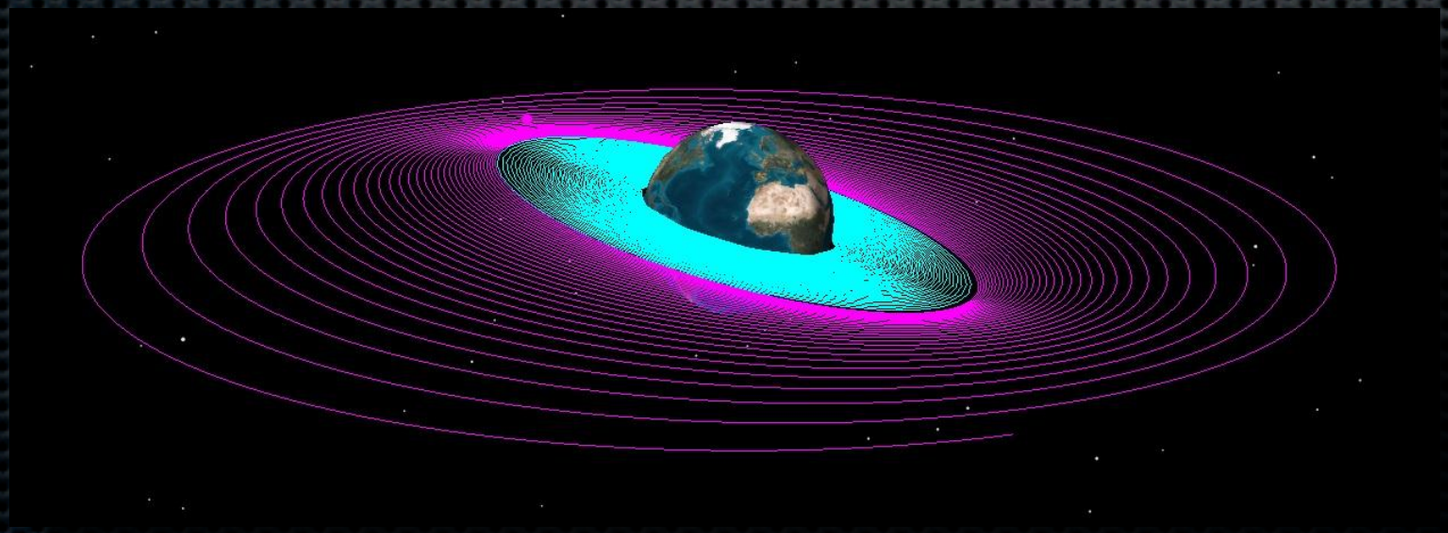
Eclipses

In the absence of batteries, engines are no longer powered by solar panels



Transfer Time

Electric orbit-raising takes a significantly longer than chemical transfer

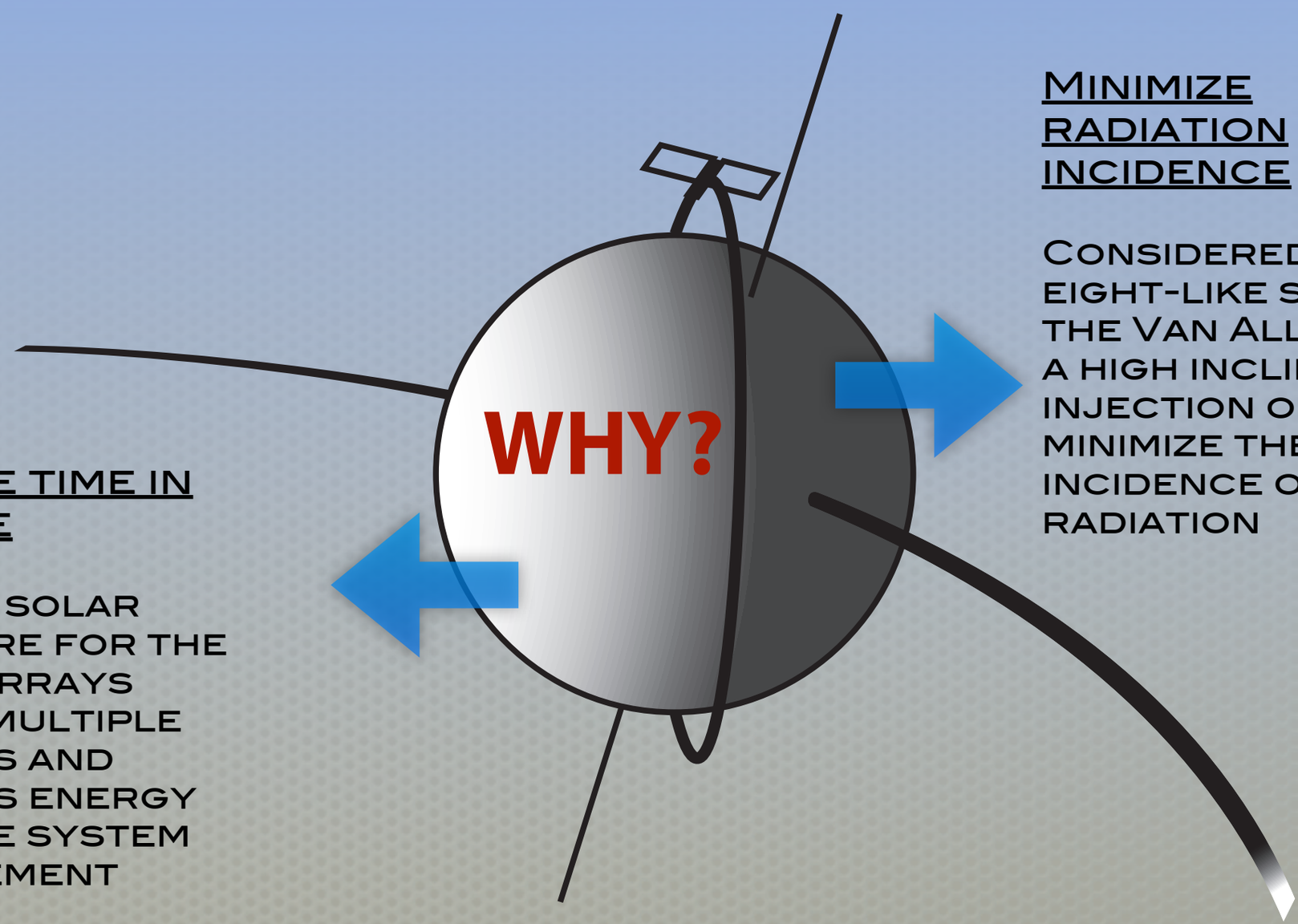


Power Requirements

Mass benefit comes at cost of high power requirements

1. Electric Orbit-Raising Problem

High-Inclination Injection Orbits



Minimize radiation incidence

Considered the eight-like shape of the Van Allen belt, a high inclination injection orbit can minimize the incidence of radiation

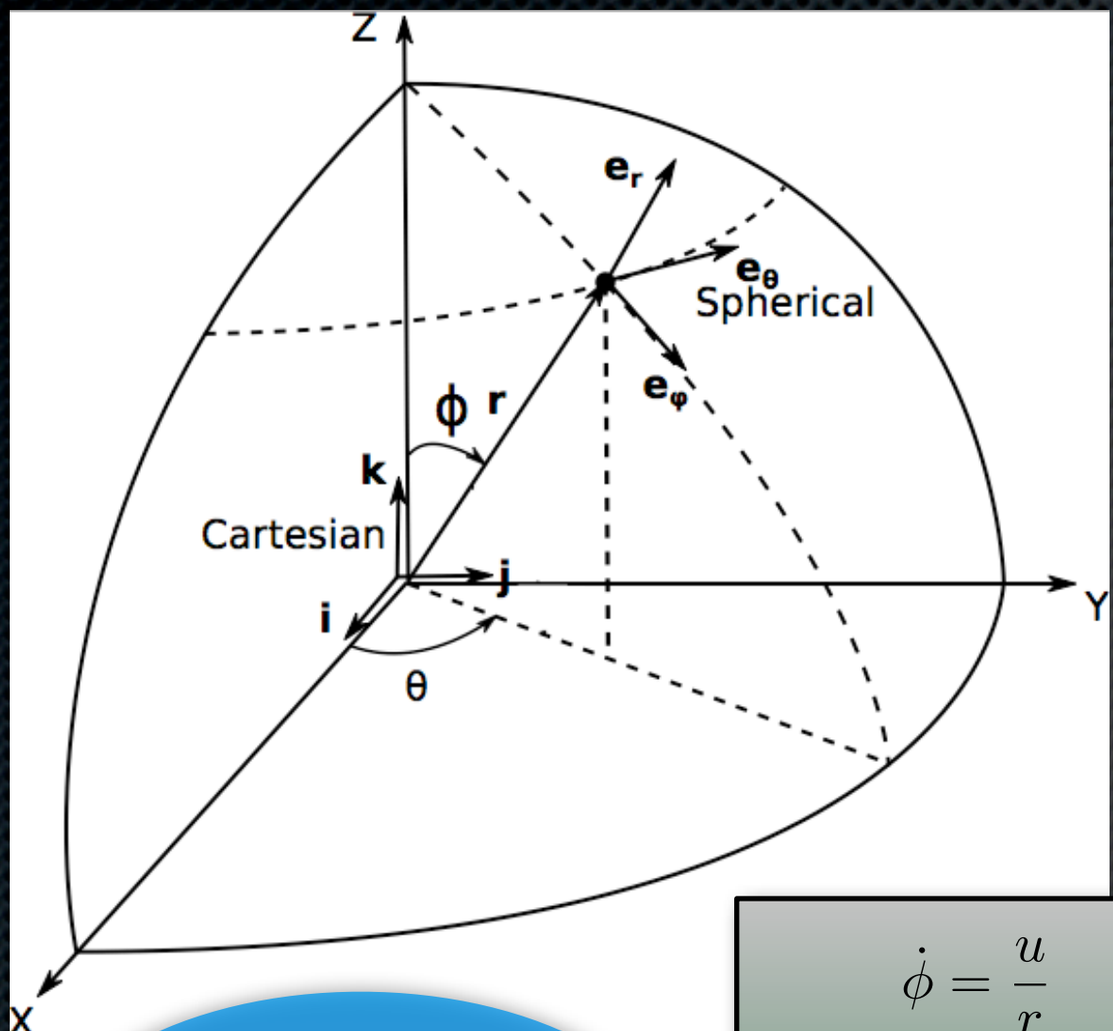
Reduce time in eclipse

Longer solar exposure for the solar arrays avoids multiple eclipses and reduces energy storage system requirement

2. Equations of Motion

Spherical Coordinates

Dutta, Libraro, Kasdin, Choueiri, 2012



θ *AZIMUTHAL ANGLE*

ϕ *POLAR ANGLE*

$$\boldsymbol{x}(t) = (\boldsymbol{r}(t), \boldsymbol{v}(t), m(t))$$ **STATE VECTOR**

$$\boldsymbol{r}(t) = (r(t), \theta(t), \phi(t))$$ **POSITION VECTOR**

$$\boldsymbol{v}(t) = (u(t), v(t), w(t))$$ **VELOCITY VECTOR**

$$(T(t), \alpha(t), \beta(t))$$ **THRUST MAGNITUDE AND DIRECTION**

$$m(t)$$ **TOTAL MASS OF THE SATELLITE**

$$\dot{\phi} = \frac{u}{r}$$

$$\dot{\theta} = \frac{v}{r \sin \phi}$$

$$\dot{r} = w$$

$$\dot{m} = -\frac{T}{c}$$

$$\dot{u} = \frac{-uw + v^2 \cot \phi}{r} + \frac{T}{m} \sin \beta$$

$$\dot{v} = -\frac{vw + uv \cot \phi}{r} + \frac{T}{m} \sin \alpha \cos \beta$$

$$\dot{w} = \frac{u^2 + v^2}{r} - \frac{\mu}{r^2} + \frac{T}{m} \cos \alpha \cos \beta$$

2. Quaternion-Based Formulation

Motivation

QUATERNIONS

- QUATERNIONS ARE BOUNDED TO BE BETWEEN -1 AND 1 → NUMERICAL ROBUSTNESS
- NONSINGULAR EQUATIONS OF MOTION → HIGH-INCLINATION INJECTION ORBITS

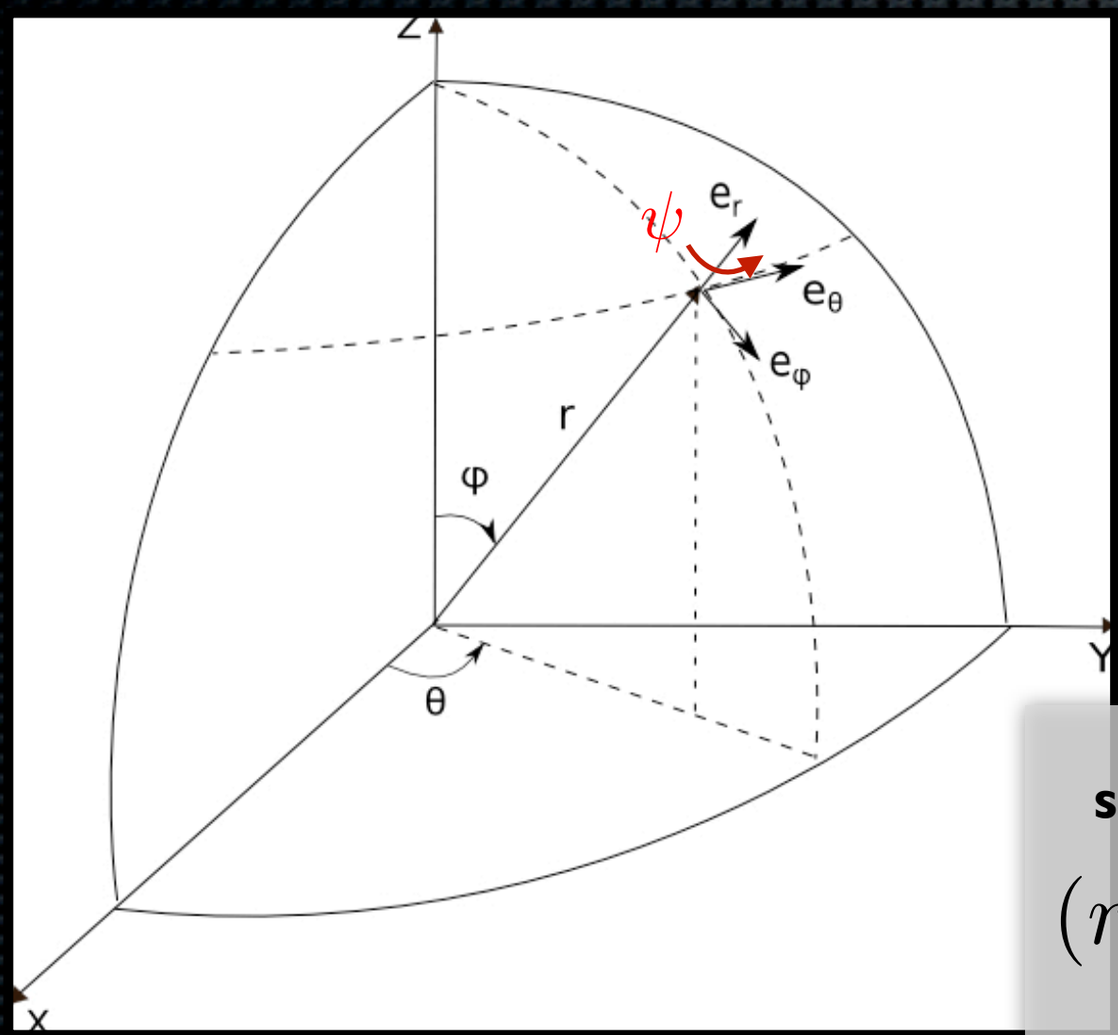
BIGGER SET OF MISSION SCENARIOS

4 CHALLENGES

- REDUCE ECLIPSE TIME
- REDUCE RADIATION INCIDENCE

2. Quaternion-Based Formulation

Mathematical Framework



Libraro, Kasdin, Dutta, Choueiri, 2014

CONSIDER A 3-2-3 ROTATION BY:

$$(\theta, \phi, \psi)$$

ASSUME THE FOLLOWING CONSTRAINT HOLDS:

$$\dot{\psi} = -\dot{\theta}\cos\phi \quad \Rightarrow \quad \omega_3 = 0$$

RADIAL VELOCITY

STATE VECTOR:

$$(r, q_1, q_2, q_3, q_4, \omega_1, \omega_2, w, m)$$

ANGULAR VELOCITY IN THE ROTATED FRAME

CONTROL VARIABLES:

$$(T, \alpha, \beta)$$

2. Quaternion-Based Formulation

Kinematics

Kane, Likins, Levinson 1983

$$[\boldsymbol{\omega}'] = [\omega_1, \omega_2, 0, 0]^T$$

START FROM

$$\boldsymbol{\omega}' = 2\boldsymbol{E}\dot{\boldsymbol{q}} \longrightarrow \dot{\boldsymbol{q}} = \frac{1}{2}\boldsymbol{E}^T\boldsymbol{\omega}'$$

APPLY THE CONSTRAINT: $\omega_3 = 0$

$$\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} q_4 & q_3 & -q_2 & -q_1 \\ -q_3 & q_4 & q_1 & -q_2 \\ q_2 & -q_1 & q_4 & -q_3 \\ q_1 & q_2 & q_3 & q_4 \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \\ 0 \\ 0 \end{pmatrix}$$

KINEMATIC EQUATIONS:

$$\dot{r} = w$$

$$\dot{q}_1 = \frac{1}{2}(q_4\omega_1 - q_3\omega_2)$$

$$\dot{q}_2 = \frac{1}{2}(q_3\omega_1 + q_4\omega_2)$$

$$\dot{q}_3 = \frac{1}{2}(-q_2\omega_1 + q_1\omega_2)$$

$$\dot{q}_4 = \frac{1}{2}(-q_1\omega_1 - q_2\omega_2)$$

2. Quaternion-Based Formulation

Equations of Motion

DYNAMICS EQUATIONS:

$$\boldsymbol{v} = \dot{r}\hat{\boldsymbol{e}}_r + r\boldsymbol{\omega} \times \hat{\boldsymbol{e}}_r$$

$$\boldsymbol{\omega} = \omega_1\hat{\boldsymbol{b}}_1 + \omega_2\hat{\boldsymbol{b}}_2 + \omega_3\hat{\boldsymbol{e}}_r$$

ACCELERATION:

$$\boldsymbol{a} = \ddot{r}\hat{\boldsymbol{e}}_{\boldsymbol{r}} + 2\dot{r}\boldsymbol{\omega} \times \hat{\boldsymbol{e}}_{\boldsymbol{r}} + r\dot{\boldsymbol{\omega}} \times \hat{\boldsymbol{e}}_{\boldsymbol{r}} + r\boldsymbol{\omega} \times \boldsymbol{\omega} \times \hat{\boldsymbol{e}}_{\boldsymbol{r}}$$

FORCE:

$$\boldsymbol{F} = F_1\hat{\boldsymbol{b}}_{\boldsymbol{1}} + F_2\hat{\boldsymbol{b}}_{\boldsymbol{2}} + F_3\hat{\boldsymbol{e}}_{\boldsymbol{r}}$$

II CONSTRAINT:

+ 2 CONSTRAINTS!

$$\omega_3 = 0$$

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$$

DYNAMICS EQUATIONS:

$$\dot{\omega}_1 = -\frac{2\dot{r}}{r}\omega_1 - \frac{F_2}{mr}$$

$$\dot{\omega}_2 = -\frac{2\dot{r}}{r}\omega_2 + \frac{F_1}{mr}$$

$$\dot{w} = r(\omega_1^2 + \omega_2^2) - \frac{\mu}{r^2} + \frac{F_3}{m}$$

3. Integration & Testing of the EOM

Matlab integration

TEST CASES

2D transfer

- **800 km ---> 10,000 km**
- **i = 90 deg**
- **T = 3 N**
- **Isp = 2,000 s**
- **m = 1,000 kg**

3D transfer

- **10,000 km ---> GEO**
- **90 deg ---> 0 deg**
- **T = 0.29 N x 4 (BPT-4000)**
- **Isp = 1,788 s**
- **m = 3,500 kg**

NONDIMENSIONAL UNITS

LU = GEO radius = 42,157 km

MU = 1,000 kg

TU = 13,716.85 s

Libraro, Kasdin, Dutta, Choueiri, 2014

3. Integration & Testing of the EOM

2D case: Polar orbit-raising

NUMERICAL ALGORITHMS:

- **NDFS: Numerical Differentiation Formulas**
- **ABM: Adams-Bashforth-Moulton**
- **RK23: Runge-Kutta (2,3)**
- **MR2: Modified Rosenbrock Formula (2nd order)**
- **TR: Trapezoidal Rule**
- **TR-BDF2: TR+Backward DFs (2nd order)**

TANGENTIAL FORCE PROFILE

$$\boldsymbol{F} = \frac{T\boldsymbol{v}}{v} = T\left(\frac{r\omega_2}{v}\hat{\boldsymbol{b}}_1 - \frac{r\omega_1}{v}\hat{\boldsymbol{b}}_2 + \frac{\dot{r}}{v}\hat{\boldsymbol{e}}_r\right)$$

TOLERANCES
RE: 10^-3
AE: 10^-6

NEW QUATERNION-BASED FORMULATION IS LESS SENSITIVE TO NUMERICAL ERROR!

Solver	NDFs	ABM	RK23	MR2	TR	TR-BDF2
Quaternions	0.3892	0.3895	0.3896	0.3898	0.3898	0.3898
Cartesian	0.5424	0.3483	0.4363	0.4357	0.4165	0.3811

RADIAL DISTANCE (GEO = 1)

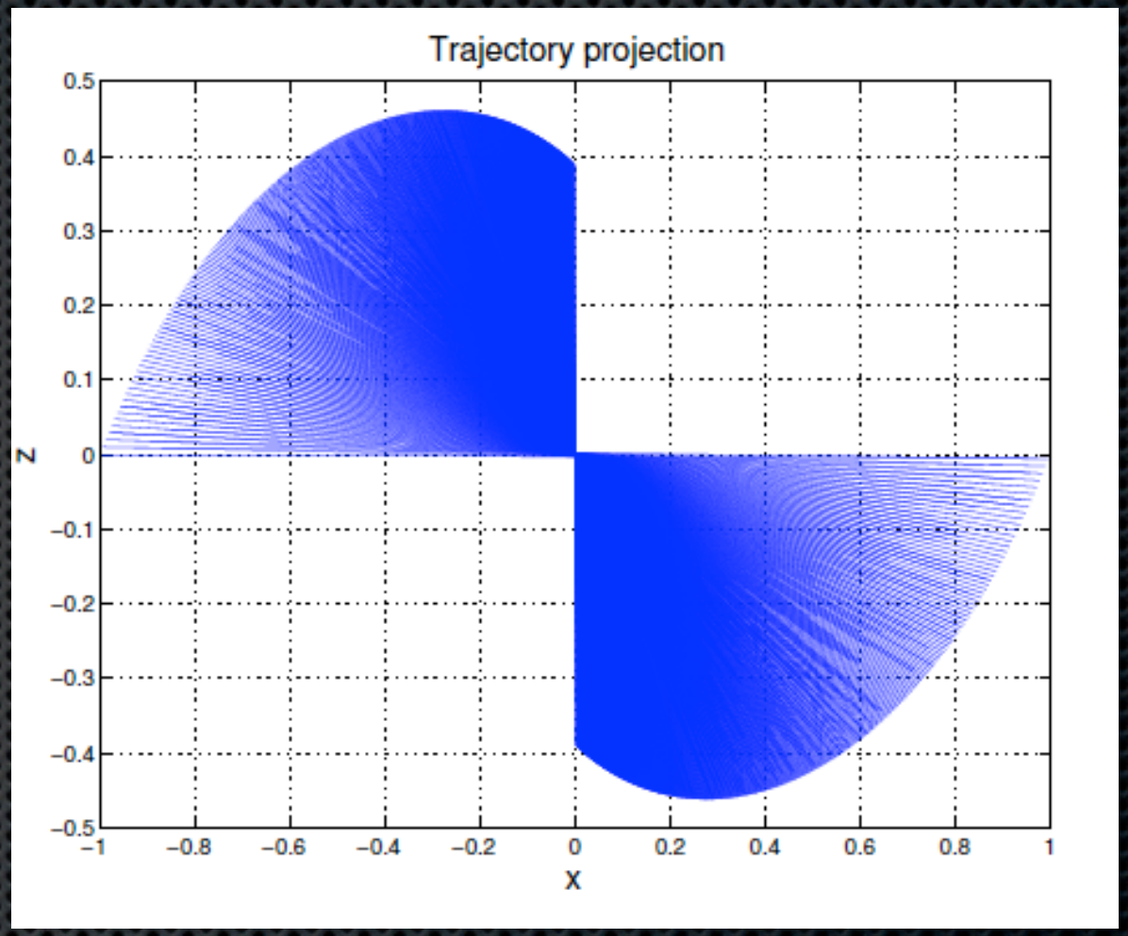
Libraro, Kasdin, Dutta, Choueiri, 2014

3. Integration & Testing of the EOM

3D case: transfer to GEO

THRUST PROFILE:

$$T(t) = \begin{cases} \cos\beta_0 \frac{\boldsymbol{v}(t)}{v(t)} + \sin\beta_0 \frac{\boldsymbol{h}(t)}{h(t)}, & \text{if } v_\perp < 0 \\ \cos\beta_0 \frac{\boldsymbol{v}(t)}{v(t)} - \sin\beta_0 \frac{\boldsymbol{h}(t)}{h(t)}, & \text{if } v_\perp > 0 \\ 0 & \text{if } v_\perp = 0 \end{cases}$$



Method: ABM
Time step: 1.58 s

Table 2 3D Orbital Transfer from a Polar Orbit to near-GEO

Coordinates	r_F	t_F	m_F	Fuel Expenditure
Quaternions	42,164 km	263.65 d	1993.0 kg	1507.0 kg
Cartesian	42,164 km	263.65 d	1993.0 kg	1507.0 kg

Libraro, Kasdin, Dutta, Choueiri, 2014

4. Trajectory Optimization

General Scheme

Mission objectives:

- Transfer time
- Fuel expenditure
- Radiation fluence

Available controls:

- Thrust magnitude
- Thrust direction

Solution techniques:

- Indirect method
- Direct method

Dynamical model:

- Spherical Earth
- J2 effect
- Injection orbit options
- Energy storage model

Problem of interest:

1. Determine optimal trajectory by minimizing mission objectives
2. Take into account dynamic constraints (EOM)
3. Consider eclipse conditions that prohibits thrusting
4. Consider available on-board energy storage

4. Trajectory Optimization

Solution Methodology

Direct method

- Discretization of the time interval

$$0 = \tau_1 < \tau_2 < \ldots < \tau_n = 1 \qquad t_k = t_f \tau_k \qquad h_k = t_k - t_{k-1}$$

- Discretization of the state and control variables

$$\boldsymbol{x}_k = \boldsymbol{x}(t_k) \qquad \boldsymbol{u}_k = \boldsymbol{u}(t_k)$$

- Defects evaluated at each segment (trapezoidal scheme)

$$\boldsymbol{\zeta}_k = \boldsymbol{x}_k - \boldsymbol{x}_{k-1} - \frac{h_k}{2}(\boldsymbol{f}_k + \boldsymbol{f}_{k-1})$$

- Parameter optimization → Use NLP (Nonlinear Programming) solver

<u>Goal:</u> minimize mission objectives by driving all defects to zero

4. Trajectory Optimization

The solver in steps

I. Set up decision variables, bounds, initial guess and constraints

II. Executes LOQO (NLP solver)

Vanderbei, 2006

III. Extracts state and control variables from final solution

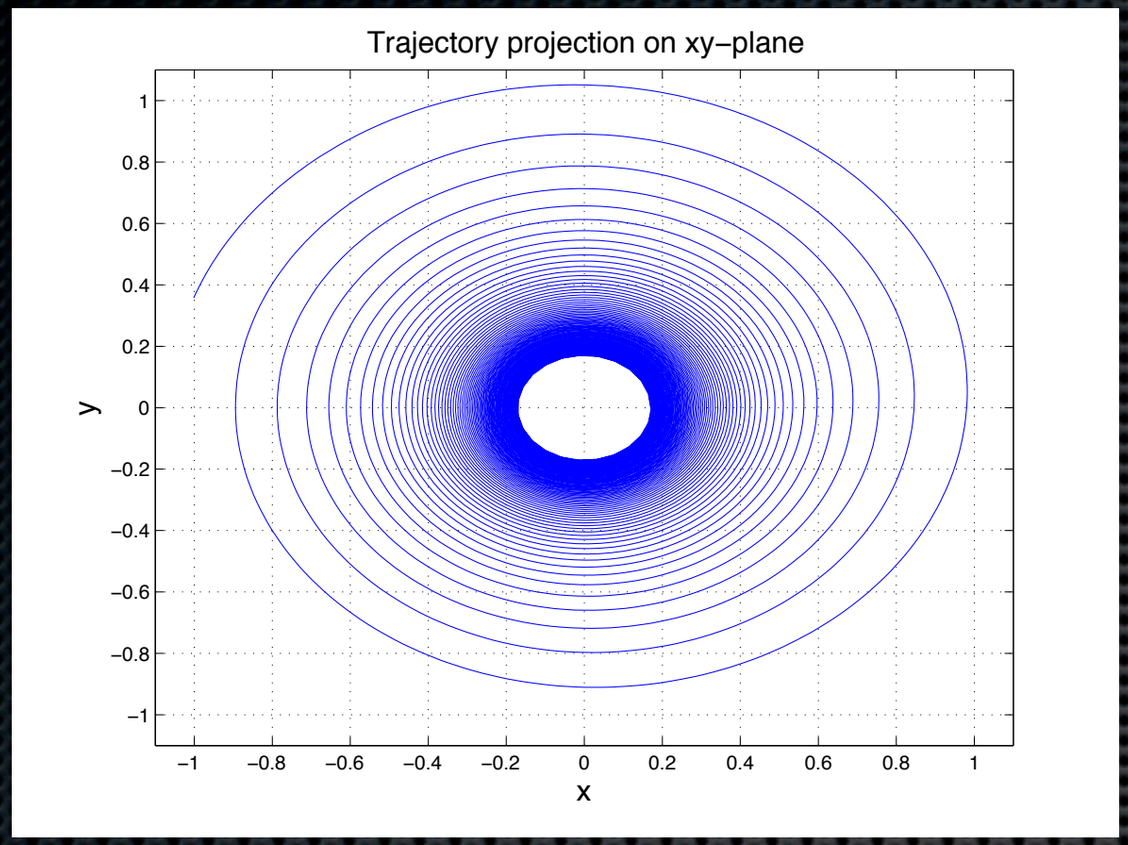
IV. Plot trajectory, state and control variables

Convergence factors

- Initial guess
- Bounds for variables
- Tolerances
- Number of nodes

5. Optimization Results

Equatorial Transfer to GEO

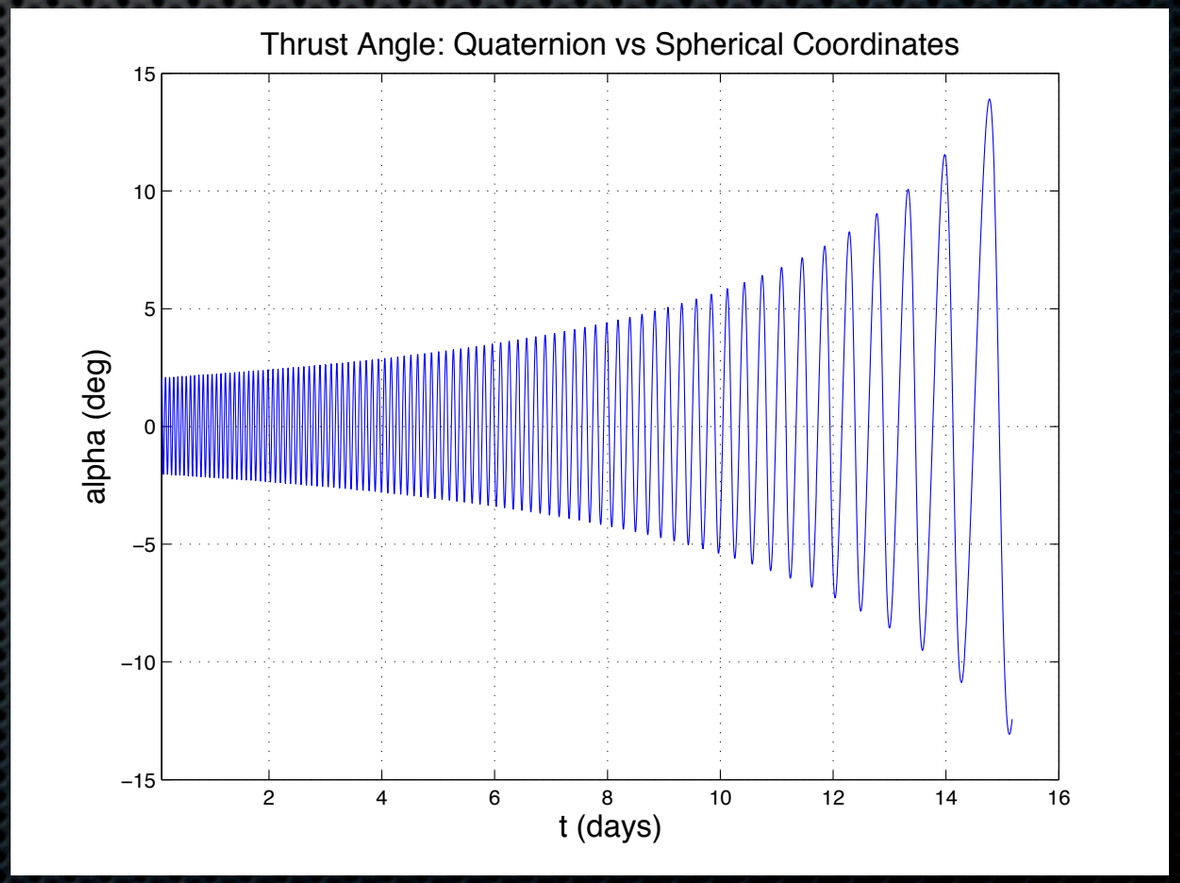


INITIAL ALTITUDE: 800 km

INITIAL MASS: 1,000 kg

THRUST: 3 N

SPECIFIC IMPULSE: 2,000 s



Thrust Angle:

- Near tangential
- Sinusoidal variation
- Amplitude increasing over time

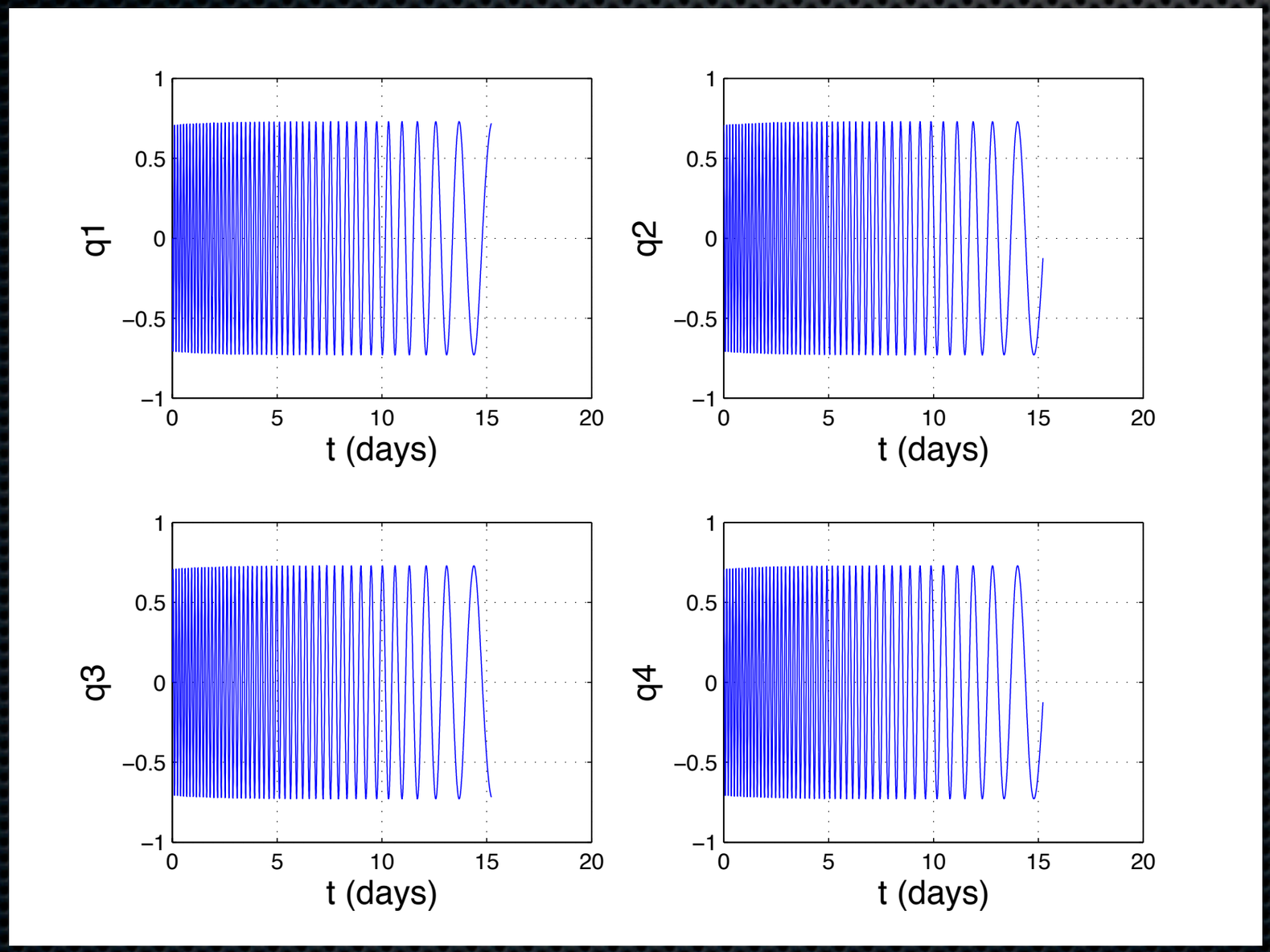
Transfer time:

t_f = 15.1702 d *(quaternions)*

t_f = 15.1703 d *(spherical)*

5. Optimization Results

Equatorial Transfer to GEO



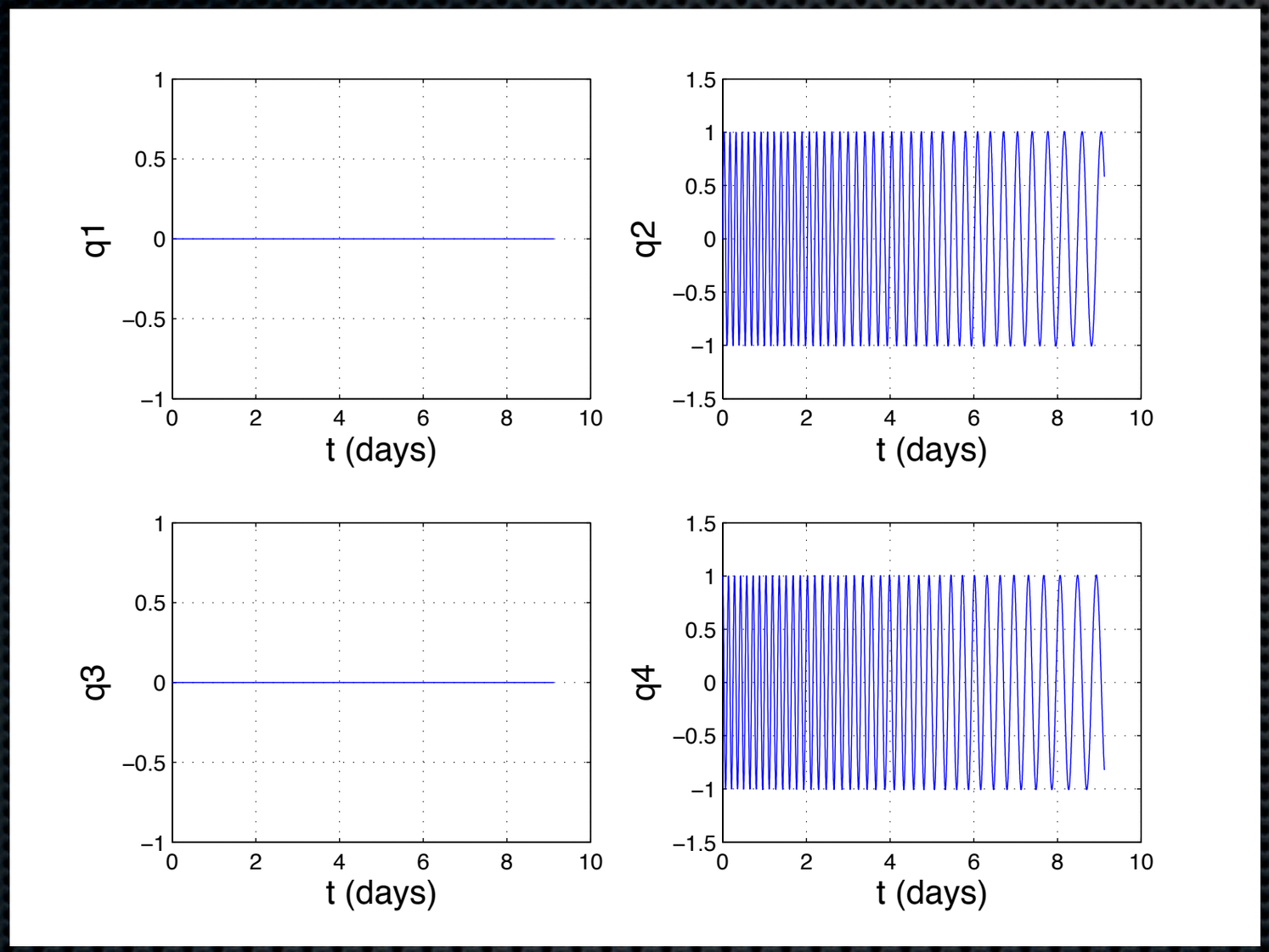
QUATERNIONS EVOLVE SINUSOIDALLY

$q_1 = -q_3$

$q_2 = q_4$

5. Optimization Results

Polar Orbit-Raising



INITIAL ALTITUDE: 800 km

FINAL ALTITUDE: 10,000 km

INITIAL MASS: 1,000 kg

THRUST: 3 N

SPECIFIC IMPULSE: 2,000 s

Transfer time:

$t_f = 9.1203\ d$

5. Optimization Results

Plane-Change Maneuver

Inclination	Final Time
0 deg	9.7 d
10 deg	19.4 d
20 deg	35.7 d
30 deg	50.6 d
40 deg	65.5 d
50 deg	79.6 d

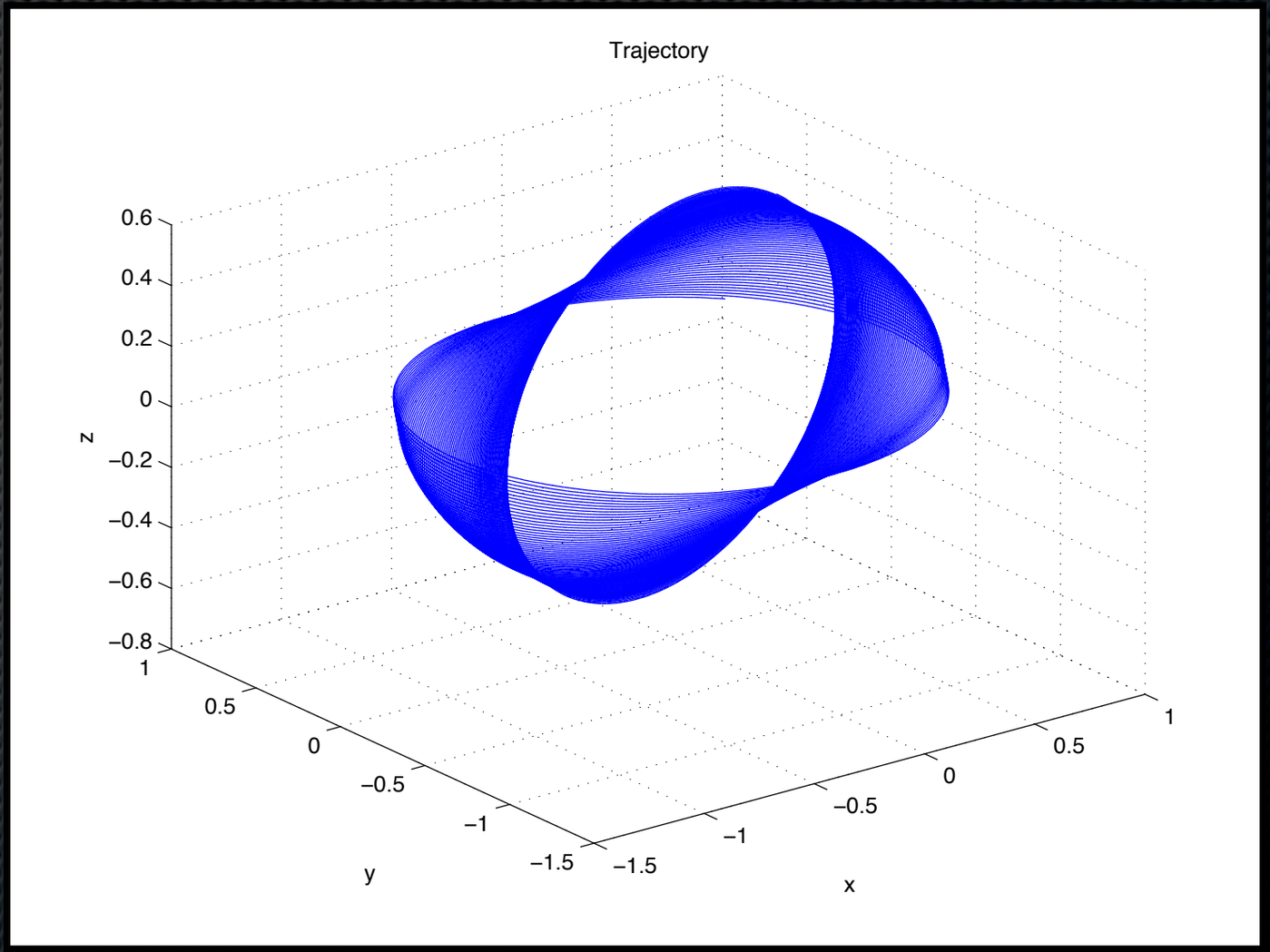
INITIAL ALTITUDE: 25,000 km

INITIAL MASS: 2,000 kg

INITIAL INCLINATION: 50 deg

THRUST: 1.16 N

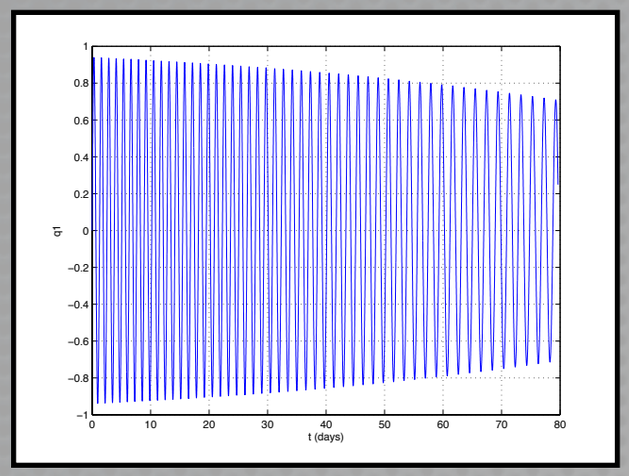
SPECIFIC IMPULSE: 1,788 s

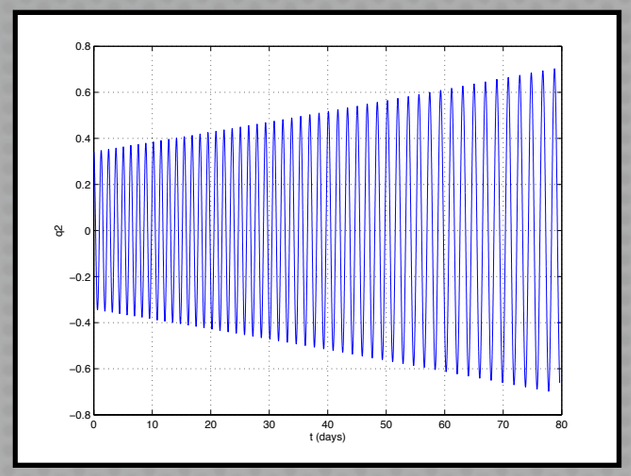


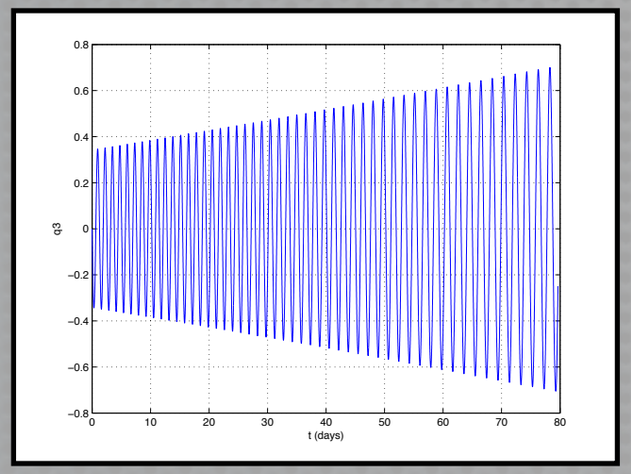
5. Optimization Results

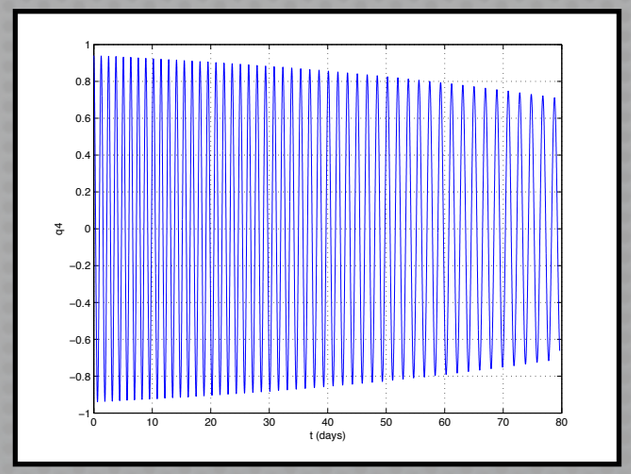
Plane-Change Maneuver

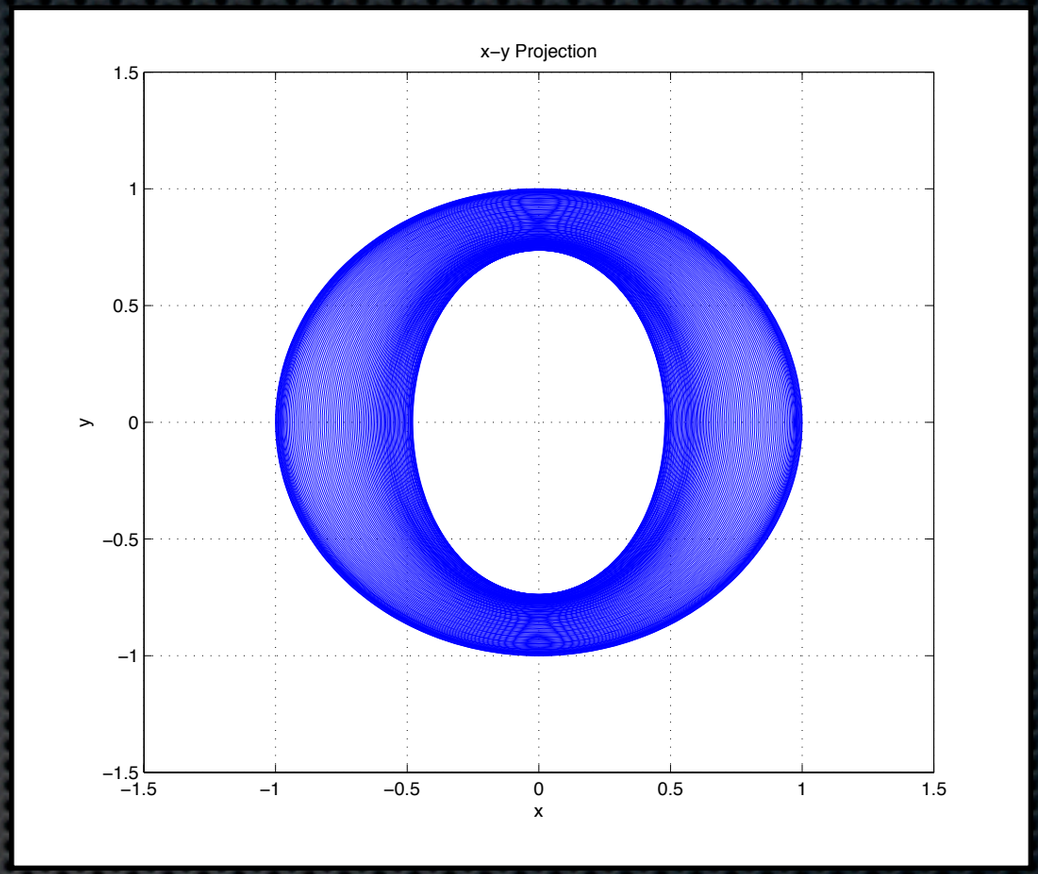
QUATERNIONS VS TIME:

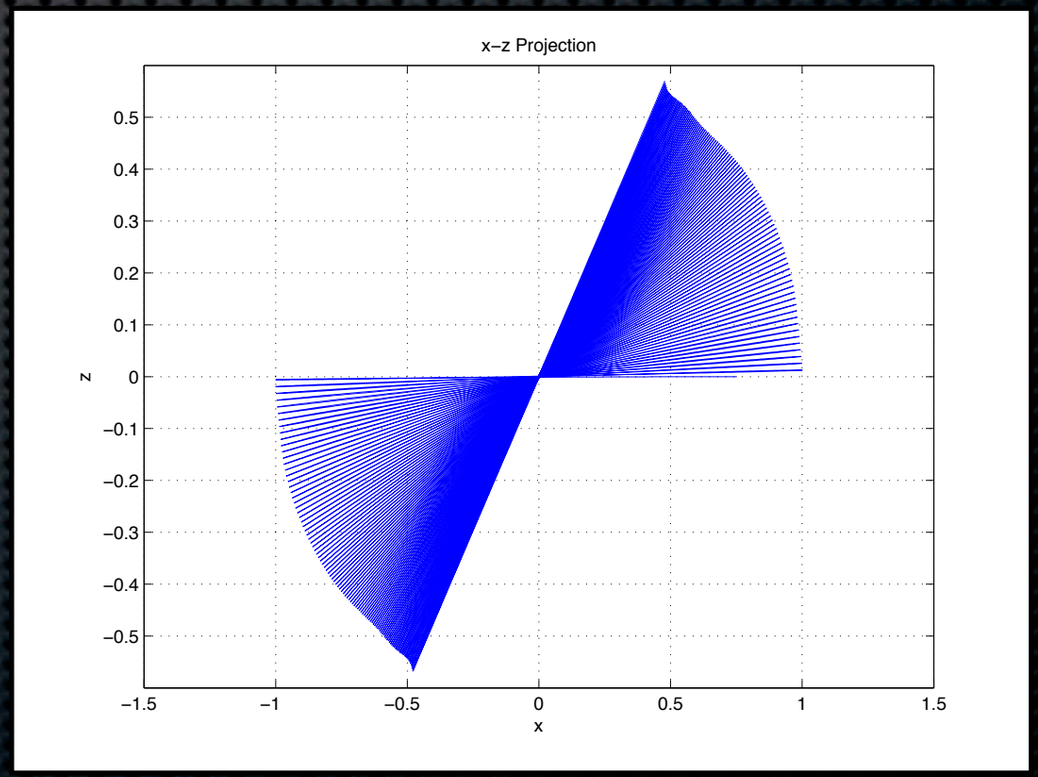












Concluding Remarks

- WE APPROACHED THE LOW-THRUST PROBLEM USING A GLOBALLY NON-SINGULAR QUATERNION-BASED FORMULATION;

- A NON-SINGULAR FORMULATION ALLOWS FOR THE INVESTIGATION OF HIGH-INCLINATION INJECTION ORBIT OPTIONS;

- HIGH-INCLINATION INJECTION ORBITS CAN POTENTIALLY ADDRESS SOME OF THE MAIN CHALLENGES FACED DURING A LOW-THRUST TRANSFER;

- THE NEW QUATERNION-BASED FORMULATION HAS BEEN TESTED AND COMPARED TO THE CARTESIAN FORMULATION PROVING TO BE LESS SENSITIVE TO NUMERICAL ERROR;

- THE MINIMUM-TIME LOW-THRUST PROBLEM HAS BEEN SOLVED FOR BOTH PLANAR AND NON-PLANAR TRANSFERS;

  - THE POLAR ORBIT-RAISING MANEUVER PROVES HOW THE NEW FORMULATION ALLOWS TO PUSH THE INCLINATION OF THE INJECTION ORBIT TO ITS UPPER LIMIT;

- FUTURE WORK INCLUDES INTRODUCING RADIATION AND PERTURBATIVE FORCES INTO THE PROBLEM IN ORDER TO INVESTIGATE THE POTENTIAL OF HIGH-INCLINATION INJECTION ORBITS AND THE APPLICABILITY OF THE NEW FORMULATION.

THANK YOU!

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